1. Attempt all problems. Show all work.

2. The problem weights are as given in the parentheses.

3. The exam is open book, open notes.

4. Laptops, calculators are allowed.

5. No communication about the exam is allowed with anybody except the proctor.

P1. A store opens for business at time zero. Male customers arrive at the store according a Poisson process with rate 10 per hour, and female customers arrive according to a Poisson process with rate 20 per hour. The two arrival streams are independent.

(a) What is the probability that the first customer is a male? (Show the details.)

(b) What is the distribution of $F(t)$, the number of female customers who arrive at the store up to time $t$?

(c) What is distribution of $M(t)$, the number of male customers who arrive at the store up to time $t$?

(d) Let $C(t)$ be the total number of customers who arrive at the store up to time $t$. What is the variance of $C(t)$?

Solution:

(a) Let $X$ be the time until the arrival of the first male customer, and $Y$ be the time until arrival of the first female customer. Then $X \sim \text{exp}(10)$ and $Y \sim \text{exp}(20)$. $X$ and $Y$ are independent. Hence the probability that the first customer is a male is given by

$$P(X < Y) = 10/(10 + 20) = 1/3.$$ 

(b) $P(20t)$.

(c) $P(10t)$.

(d) $C(t) = F(t) + M(t)$, sum of two independent Poisson rvs. Hence $C(t) \sim P(30t)$. Hence $\text{Var}(C(t)) = 30t$. 

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P2. A single server handles calls that arrive according to a PP with rate 10 per hour. If a caller finds the server idle, she immediately starts getting served, and the call lasts for an exponential time with mean five minutes. If an incoming caller finds the server busy, she goes away and is permanently lost. Successive service times are iid. The server shift is from from 8am to 12 noon, and there are no callers in the system at 8am. If the server is idle at noon, he stops working at noon. If he is busy at noon, he completes handling the current caller, and then stops working.

(a) What is the probability that the server is delayed beyond his scheduled departure?
(b) What is the expected departure time of the server?
(c) What is the expected time the server is busy handling calls until he departs?

Solution:

(a) Let $X(t)$ be 1 if the server is busy at time $t$ hours after 8am, and 0 otherwise. Then for $0 \leq t \leq 4$, \{ $X(t), t \geq 0$ \} is a CTMC on state space \{0, 1\} of Example 4.3 with $\lambda = 10$ and $\mu = 12$. The probability that the server is delayed beyond his scheduled departure is

$$P(X(4) = 1|X(0) = 0) = p_{0,1}(4) = (10/22)(1 - e^{-22 \times 4}) = .4545$$

(b) The server leaves at noon with probability .5454. With probability .4545 he stays on to complete the current customer, which takes additional five minutes. Hence the expected departure time is 2.27 minutes after 12noon.

(c) The expected time the server is busy after 4 hours is (Expected remaining service time of the customer given that the server is busy at time 4)*(Probability that the server is busy at time 4) = (1/12)*.4545 = .0207. Hence the total busy time = 1.7975 + .0207 = 1.818 hours.

P3. There are $k$ numbered parking spaces outside a 24-hour restaurant, with the first space being the closest and space $k$ being the farthest. Customer cars arrive according to a PP($\lambda$) and occupy the closest available space (customers being too lazy to walk more than absolutely necessary!) If all spaces are busy the arriving customers are lost. A customer stays in the restaurant for an exp($\mu$) amount of time and then leaves. The customer service times are iid. Let $X_i(t)$ be number of cars in the ith space at time $t$. Let $Z_i(t)$ be the total number of cars in the spaces 1, 2, \cdots, $i$ at time $t$, ($1 \leq i \leq k$).
(a) Is \( \{X_i(t), t \geq 0\} \) a CTMC, \( 1 \leq i \leq k \)? If yes, give its rate matrix. If not, state the reasons.

(b) Write \( Z_i(t) \) (\( 1 \leq i \leq k \)) in terms of the other \( X_j(t) \)'s.

(c) Is \( \{Z_i(t), t \geq 0\} \) a CTMC, \( 1 \leq i \leq k \)? If yes, give its rate matrix. If not, state the reasons.

(d) Compute the limiting distribution of \( Z_i(t), 1 \leq i \leq k \).

(e) Let \( m_j \) be the mean of \( Z_j(t) \) in steady state, \( 1 \leq i \leq k \). Using the \( m_j \)'s, show how to compute \( x_i \), the mean of \( X_i(t) \) in steady state.

(f) Compute the limiting probability that the \( i \)th space is occupied, in terms of \( x_i \).

Solution:

(a) \( \{X_1(t), t \geq 0\} \) is a CTMC, as in Example 4.3. \( \{X_i(t), t \geq 0\} \) for \( i \geq 2 \) is not a CTMC, since the arrival event to the \( i \)th space depends on the state of the spaces \( \{1, 2, \cdots, i - 1\} \).

(b) \( Z_i(t) = X_1(t) + X_2(t) + \cdots X_i(t) \).

(c) \( \{Z_i(t), t \geq 0\} \) is a birth and death on \( \{0, 1, \cdots, i\} \) with birth rates \( \lambda = \lambda_j, \ 0 \leq j < i \), and death rates \( \mu_j = j \mu, \ 0 \leq j \leq i \).

(d) Use Example 4.23. Let

\[
\rho_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j} = \frac{1}{j!} (\lambda/\mu)^j, \ 0 \leq j \leq i.
\]

Hence

\[
p_j = \lim_{t \to \infty} P(Z_i(t) = j) = \frac{\rho_j}{\sum_{k=0}^i \rho_k}, \ 0 \leq j \leq i.
\]

We can compute

\[
m_i = \sum_{j=0}^i j \rho_j.
\]

(e) \( X_i(t) = Z_i(t) - Z_{i-1}(t). \) Hence

\[
x_i = m_i - m_{i-1}, \ 1 \leq i \leq k
\]

(with \( m_0 = 0 \).)

(f) \( x_i = \lim_{t \to \infty} [P(X_i(t) = 1) * 1 + P(X_i(t) = 0) * 0] = \lim_{t \to \infty} P(X_i(t) = 1).
\]

Thus \( x_i \) also equals the limiting probability that the \( i \)th space is occupied.