Problem 1. Jobs in a manufacturing shop need to go through two tasks: Task 1 and Task 2. There are two workers in this shop: Worker 1 is responsible from Task 1 and Worker 2 is responsible from Task 2. Each job needs to be first processed by Worker 1 then by Worker 2. Two workers cannot work on the same job at the same time. If worker $i$ is working on a job at the beginning of an hour, then she will complete that task by the end of the hour with probability $p_i$, independent of everything else, where $i = 1, 2$.

At any given time during the manufacturing process, there can be exactly two jobs in the shop. A job that completes both tasks leave the shop and is immediately replaced by a new job.

Workers never stay idle except for two cases: If both jobs in the shop finish Task 1 but require Task 2, then there is no work available for Worker 1 and hence she stays idle. If neither of the jobs in the shop is done with Task 1, then there is no work available for Worker 2 and hence she stays idle.

We will model this system as a discrete-time Markov chain (DTMC).

(a) Define the state of the system explicitly. Provide the state space.

(b) Obtain the transition probability matrix.

(c) Justify that the stochastic process that you defined in parts (a) and (b) is a DTMC.
Problem 2. Consider the DTMC \( \{X_n, n \geq 0\} \) with state space \( S = \{0, 1, 2, 3, 4, 5\} \) and transition probability matrix

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0.2 & 0 & 0 & 0 & 0.8 & 0 \\
0 & 0.7 & 0 & 0 & 0 & 0.3 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}.
\]

(a) Is this chain irreducible? Why/why not?

(b) Obtain the period of each state.

(c) Does a unique occupancy distribution exist? Why/why not? If it exists, explain in words how you can obtain it.

(d) Does a unique stationary distribution exist? Why/why not? If it exists, explain in words how you can obtain it.

(e) Does a unique limiting distribution exist? Why/why not? If it exists, explain in words how you can obtain it.

Problem 3. Consider the DTMC defined in Problem 2. Suppose that this DTMC starts in state 0. Obtain the following. (You do not need to provide a numerical answer.)

(a) The probability that states 4 and 5 are visited in periods 4 and 5, respectively.

(b) \( E[X_4] \)

(c) The expected number of times state 0 is visited during periods \( \{0, 1, \ldots, 4\} \).

(d) The expected total cost during periods \( \{0, 1, \ldots, 4\} \) if each visit to state \( i \) costs \( i \) dollars for \( i \in S \).

Problem 4. Consider the DTMC \( \{X_n, n \geq 0\} \) with state space \( S = \{0, 1, 2, 3\} \) and transition probability matrix

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 \\
p & 0 & 1-p & 0 \\
0 & p & 0 & 1-p \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

(a) What is the long-run fraction of time that the DTMC spends in state 0?

(b) Suppose that the system gains \((i + 1)\) units of reward each time the DTMC transits from state \( i \) to state \( i + 1 \) for \( i = 0, 1, 2 \). What is the long-run average units of reward gained from this system?

(c) If the DTMC starts in state 2, what is the expected time it takes to visit state 0 for the first time?