Question 1)

Jobs in a manufacturing shop need to go through two tasks: Task 1 and Task 2. There are two workers in this shop. Worker 1 is responsible for Task 1 and Worker 2 is responsible for Task 2. Each job needs to be first processed by Worker 1 and then by Worker 2. Two workers cannot work on the same job at the same time. If worker $i$ is working on a job at the beginning of an hour, then she will complete that task by the end of the hour with probability $p_i$, independent of everything else, where $i = 1, 2$.

At any given time during the manufacturing process, there can be exactly two jobs in the shop. A job that completes both tasks leave the shop and is immediately replaced by a new job.

Workers can stay idle except for two cases: If both jobs in the shop finish Task 1 but require Task 2, then there is no work available for Worker 1 and hence she stays idle. If neither of the jobs in the shop is done with Task 1, then there is no work available for Worker 2 and hence she stays idle.

a) Define the state of the system explicitly. Provide the state space.

Let $X_n$ be the number of items that require both tasks at the end of day $n$. Then, the state space is $S = \{0, 1, 2\}$

b) Obtain the one-step transition matrix.

The one-step transition matrix is given as follows:

$$
P = \begin{bmatrix}
(1 - p_2) & p_2 & 0 \\
p_1(1 - p_2) & p_1p_2(1 - p_1)(1 - p_2) & (1 - p_1)p_2 \\
0 & p_1 & (1 - p_1)
\end{bmatrix}
$$

c) Justify that the stochastic process you defined in parts (a) and (b) is a DTMC.

We already defined the state space and the one-step transition matrix, which is stochastic. Therefore, to justify that a stochastic process is a DTMC, checking whether it satisfies the Markov Property is sufficient.

If $X_n = 0$,

$$
X_{n+1} = \begin{cases} 
X_n + 1 & \text{with probability } p_2 \\
X_n & \text{with probability } (1 - p_2)
\end{cases}
$$

If $X_n = 1$,

$$
X_{n+1} = \begin{cases} 
X_n - 1 & \text{with probability } p_1(1 - p_2) \\
X_n & \text{with probability } p_1p_2 + (1 - p_1)(1 - p_2) \\
X_n + 1 & \text{with probability } (1 - p_1)p_2
\end{cases}
$$
If $X_n = 2$,

$$X_{n+1} = \begin{cases} X_n - 1 & \text{with probability } p_1 \\ X_n & \text{with probability } (1 - p_1) \end{cases}$$

Therefore, we can conclude that $X_{n+1}$ only depends on $X_n$. Thus, the stochastic process satisfied the Markov Property and hence, it is a DTMC.

Question 2)

Consider the DTMC $\{X_n, n \geq 0\}$ with state space $S = \{0, 1, 2, 3, 4, 5\}$ and the following one-step transition probability matrix.

$$P = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

a) **Is this chain irreducible? Why/why not?**

Yes. It is irreducible since all states communicate with each other.

b) **Obtain the period of each state.**

Starting from state 0, you can return to state 0 in periods $\{4, 8, 12, \ldots\}$. Therefore, the period of state 0 is 4. Since the chain is irreducible, the periods of all states are 4.

c) **Does a unique occupancy distribution exist? Why/why not? If it exists, explain in words how you can obtain it.**

Since the DTMC has finite state space and is irreducible, the unique occupancy distribution can be obtained by solving balance equations.

d) **Does a unique stationary distribution exist? Why/why not? If it exists, explain in words how you can obtain it.**

Since the DTMC has finite state space and is irreducible, the unique stationary distribution can be obtained by solving balance equations.

e) **Does a unique limiting distribution exist? Why/why not? If it exists, explain in words how you can obtain it.**

Since the DTMC is periodic, a unique limiting distribution does not exist.
Question 3)
Consider the DTMC defined in Question 2. Suppose that his DTMC starts in state 0. Obtain the following. (You do not need to provide a numerical answer.)

a) The probability that states 4 and 5 are visited in periods 4 and 5, respectively.
   \[ P(X_5 = 5, X_4 = 4 \mid X_0 = 0) = P(X_5 = 5 \mid X_4 = 4, X_0 = 0)P(X_4 = 4 \mid X_0 = 0) \]
   \[ = P(X_5 = 5 \mid X_4 = 4)P(X_4 = 4 \mid X_0 = 0) = p_{4,5}p_{0,4}. \]

b) \( E[X_4] \)
   Since we know that we are in state 0 at time 0, \( p_{0,i}^{(4)} \) is the probability of being in state \( i \) in period 4. Therefore,
   \[ E[X_4] = \sum_{i=0}^{5} (i \cdot p_{0,i}^{(4)}) \]

c) The expected number of times state 0 is visited during periods \( \{0, 1, 2, 3, 4\} \).
   The matrix \( M(4) = \sum_{i=0}^{5} P^i \) gives us the occupancy matrix where \( M(4)_{i,j} \) represents the expected number of visits to state \( j \) in the first 4 periods (including period 0) given that we \( X_0 = i \). Therefore, \( M(4)_{0,0} \) gives us the expected number of times state 0 is visited during the first 4 periods (including period 0).

d) The expected total cost during periods \( \{0, 1, 2, 3, 4\} \) if each visit to state \( i \) costs \( i \) dollars \( \forall i \in S \).
   Assume \( M(4) \) as in part c). Then,
   \[ E[\text{Total Cost}] = \sum_{i=0}^{5} (i \cdot M(4)_{i,0}). \]

Question 4)
Consider the DTMC \( \{X_n, n \geq 0\} \) with state space \( S = \{0, 1, 2, 3\} \) and the following one-step transition probability matrix.

\[ P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ p & 0 & 1-p & 0 \\ 0 & p & 0 & 1-p \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

a) What is the long-run fraction of time that DTMC spends in state 0?
   The DTMC has finite state space and is irreducible. Therefore, the limiting occupancy distribution can be obtained by solving the following balance equations:
   \[ \pi_0 = p\pi_1 \]
   \[ \pi_1 = \pi_0 + p\pi_2 \]
   \[ \pi_2 = (1-p)\pi_1 + \pi_3 \]
   \[ \pi_3 = (1-p)\pi_2 \]
   After solving these equations, we can find that \( \pi_0 = \frac{p^2}{1+p^2+(1-p)^2} \) is the long-run fraction
of time that DTMC spends in state 0.

b) **Suppose that the system gains** \((i + 1)\) **units of reward each time the DTMC transits from state** \(i\) **to state** \(i + 1\) **for** \(i = 0, 1, 2\). **What is the long-run average units of reward gained from this system?**
Take any \(i = 0, 1, 2\). The long-run fraction of transitions from state \(i\) to state \(i + 1\) is \(\pi_ip_{i,i+1}\). Then,
\[
E[\text{reward}] = \sum_{i=0}^{2} (i + 1)\pi_ip_{i,i+1} = \pi_0 + 2(1-p)\pi_1 + 3(1-p)\pi_2.
\]

c) **If the DTMC starts in state 2, what is the expected time it takes to visit state 0 for the first time?** Define \(m_i\) as the expected time it takes to visit state 0 given that the DTMC starts at state \(i\). Using the Theorem 2.13 of the textbook, we need to solve the following equations:
- \(m_0 = 0\)
- \(m_1 = 1 + (1-p)m_2\)
- \(m_2 = 1 + pm_1 + (1-p)m_3\)
- \(m_3 = 1 + m_2\)
Solving these equations simultaneously yields \(m_2 = \frac{2}{p^2}\) as the expected time it takes to visit state 0 given that the DTMC starts at state 2.