The statement is true for \( k = 1 \) by definition of transition probabilities. Suppose it is true for \( k \). We shall show that it is true for \( k + 1 \), thus proving it in general by induction. We have
\[
P(X_{k+1} = i_{k+1}, \ldots, X_1 = i_1 | X_0 = i_0)
= P(X_{k+1} = i_{k+1} | X_k = i_k, \ldots, X_1 = i_1, X_0 = i_0) 
\times P(X_k = i_k, \ldots, X_1 = i_1 | X_0 = i_0)
= P(X_{k+1} = i_{k+1} | X_k = i_k) 
\times P(X_k = i_k, \ldots, X_1 = i_1 | X_0 = i_0)
= p_{i_{k-1}, i_k} p_{i_k, i_{k-1}} \cdots p_{i_0, i_1}.
\]
Here the second equality follows due to Markov property, and the third one from the induction hypothesis.

2. Conceptual Problem 2.2.
Consider the following DTMC on the state space \( S = \{1, 2, 3\} \) with the transition probability matrix
\[
P = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]
and the initial distribution \((a_1, a_2, a_3)\). Then the pmf of \( X_1 \) is \((a_3, a_1, a_2)\), and that of \( X_2 \) is \((a_2, a_3, a_1)\). Using this we get,
\[
P(X_1 = 1, X_2 = 2, X_3 = 3) = P(X_3 = 3 | X_2 = 2)P(X_2 = 2 | X_1 = 1)P(X_1 = 1) = a_3,
\]
whereas
\[
P(X_2 = 1, X_3 = 2, X_4 = 3) = P(X_4 = 3 | X_3 = 2)P(X_3 = 2 | X_2 = 1)P(X_2 = 1) = a_2.
\]
In general, the two probabilities are not equal to each other unless \( P(X_1 = i) = P(X_2 = i) \).

3. Computational Problem 2.4.
The distribution in week 10 (starting with brand \( A \) in week 1) is given by
\[
a^{(10)} = [1 0 0] \ast P^{9} = [.1977 \ .3140 \ .4884].
\]
Using the $P$ matrix of Example 2.4, the desired probability is obtained as
\
\[ P(X_4 = 3, X_3 = 5, X_2 = 2, X_1 = 4 | X_0 = 5) = p_{5,4} p_{4,2} p_{2,5} p_{5,3} = (.1494)(.2240)(.9502)(.2240) = 0.0071. \]

Using the $P$ matrix of Example 2.5, the desired probability is obtained as
\
\[ P(X_4 = 2, X_2 = 0, X_1 = 0 | X_0 = 0) = p_{0,0}^2 p_{0,2}^{(2)} = b^2 c. \]