1. **Computational Ex. 8.54.**

(a) Let \( S_n \) be the time of the \( n \)-th purchase when the customer pays \( c \). (Note that this does not include the free replacements. The successive failures form a \( \text{PP}(\lambda) \), and hence \( \{X_n = S_n - S_{n-1}, n \geq 1\} \) forms a sequence of iid random variables, with common mean \( 1 + 1/\lambda \). Let \( Z_n = c \), the amount the customer pays at time \( S_n \). Let \( N(t) \) be the new purchases by a customer up to time \( t \). Thus \( \{N(t), t \geq 0\} \) is an RP generated by \( \{X_n, n \geq 1\} \). We have

\[
Z(t) = \sum_{n=1}^{N(t)} Z_n,
\]

and \( \{(X_n, Z_n), n \geq 1\} \) are iid bi-variate random variables. Hence \( \{Z(t), t \geq 0\} \) is a renewal reward process.

Next let \( N_f(t) \) be the number of failures up to time \( t \) and define

\[
Y_n = c - d(N_f(S_n) - N_f(S_{n-1})).
\]

Thus \( Y_n \) is the profit during the \( n \)-th cycle. Since \( \{N_f(t), t \geq 0\} \) is a \( \text{PP}(\lambda) \), we see that \( \{(X_n, Y_n), n \geq 1\} \) is a sequence of iid bivariate random variables. Hence

\[
Y(t) = \sum_{n=1}^{N(t)} Y_n,
\]

is a renewal reward process.

(b) We have \( \mathbb{E}(X_n) = 1 + 1/\lambda \), \( \mathbb{E}(Z_n) = c \), and \( \mathbb{E}(Y_n) = c - d\lambda(1 + 1/\lambda) = c - d - d\lambda \). Hence, the expected cost per year to the customer is given by

\[
\lim_{t \to \infty} \frac{Z(t)}{t} = \mathbb{E}(Z_n)/\mathbb{E}(X_n) = \frac{\lambda c}{1 + \lambda},
\]

and the expected profit per year to the producer is given by

\[
\lim_{t \to \infty} \frac{Y(t)}{t} = \mathbb{E}(Y_n)/\mathbb{E}(X_n) = \frac{(c - d)\lambda - d\lambda^2}{1 + \lambda}.
\]

2. **Conceptual Ex. 8.20.** We have

\[
\phi(i) = \mathbb{E} \left[ \int_0^\infty e^{-\alpha t} C(X(t)) dt | X(0) = i \right] dt
\]

\[
= \sum_{k \in \mathbb{S}} \int_0^\infty \mathbb{E} \left[ \int_0^y e^{-\alpha t} C(X(t)) dt + \int_y^\infty e^{-\alpha t} C(X(t)) dt | Y_1 = k, S_1 = y, X(0) = i \right] dG_{ik}(y)
\]

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3. Modeling Ex. 9.8. Let $X(t)$ be 0 if the machine is down and waiting for repair, 1 if the machine is under repair, and 2 if the machine is up. We shall show that \{X(t), t \geq 0\} is an MRGP. Suppose at time a repair has just completed, i.e., $X(0) = i$. Let $X_n$ process into state 1 or 2, and $S_n$ be the time of servers implies that \{X(t), t \geq 0\} is stochastically identical to \{X(t), t \geq 0\} given $X_0 = i$ and is conditionally independent of \{X(u) : 0 \leq u < S_n, X_n = i\} given $X_n = i$. Hence \{X(t), t \geq 0\} is an MRGP.

4. Modeling Ex. 9.11. Let $S_0 = 0$, $S_n$ be the time when the $n$th server departs, and $X_n = X(S_n +)$. Let $A_n$ be the arrivals over $(S_n, S_{n+1})$. The $\{A_n, n \geq 0\}$ are iid. We have

$$X_{n+1} = \text{Bin}(X_n + A_n, 1 - \alpha).$$

hence $\{X_n, n \geq 0\}$ is a DTMC. Also, $\{Y_n = S_n - S_{n-1}n \geq 1\}$ are iid. Hence $\{(X_n, S_n), n \geq 0\}$ is an MRS embedded in the process $\{X(t), t \geq 0\}$. Also, the memoryless property of the PP and the iid nature of the inter-arrival times of servers implies that \{X(t + S_n), t \geq 0\} given $X(u) : 0 \leq u < S_n, X_n = i$ is stochastically identical to $\{X(t), t \geq 0\}$ given $X_0 = i$.
5. **Computational Ex. 9.4.** Let $S_n$ be the $n$th transition time in the CTMC \{X(t), t \geq 0\}, $Y_n = X(S_n+)$. We know that \{(Y_n, S_n), n \geq 0\} is an embedded Markov Renewal Sequence with kernel
\[
G(x) = \begin{bmatrix} 0 & 1-e^{-\lambda x} \\ 1-e^{-nx} & 0 \end{bmatrix}.
\]
Let $T_j(t)$ be the time spent in state $j$ over $(0, t)$. We are asked to compute $H_{i,j}(t) = E\{T_j(t) | X(0) = i\}$. We have
\[
E\{T_j(t) | X(0) = i, Y_1 = k, S_1 = y\} = \begin{cases} \delta_{i,j}y + H_{k,j}(t-y) & \text{if } y \leq t \\ \delta_{i,j}t & \text{if } y > t \end{cases}
\]
Unconditioning we get the following Markov Renewal Type Equation
\[
H(t) = D(t) + G \ast H(t)
\]
where
\[
D(t) = \begin{bmatrix} (1-e^{-\lambda t})/\lambda & 0 \\ 0 & (1-e^{-\mu t})/\mu \end{bmatrix}.
\]
Taking LSTs, we get
\[
\tilde{H}(s) = (I - \tilde{G}(s))^{-1} \tilde{D}(s).
\]
Using
\[
\tilde{G}(s) = \begin{bmatrix} 0 & \lambda/(\lambda + s) \\ \mu/(\mu + s) & 0 \end{bmatrix},
\]
and
\[
\tilde{D}(s) = \begin{bmatrix} 1/(\lambda + s) & 0 \\ 0 & 1/(\mu + s) \end{bmatrix},
\]
and simplifying, we get
\[
\tilde{H}(s) = \frac{1}{s(s + \lambda + \mu)} \begin{bmatrix} \mu + s & \lambda \\ \mu & \lambda + s \end{bmatrix}.
\]
Inverting this we get (using $l = \lambda$ and $m = \mu$)
\[
H_{0,0}(t) = \frac{\lambda}{(\lambda + \mu)^2}((\lambda + \mu)t + 1 - e^{-(\lambda+\mu)t}),
\]
\[
H_{0,1}(t) = \frac{\lambda}{(\lambda + \mu)^2}((\lambda + \mu)t - 1 + e^{-(\lambda+\mu)t}),
\]
\[
H_{1,0}(t) = \frac{\mu}{(\lambda + \mu)^2}((\lambda + \mu)t - 1 + e^{-(\lambda+\mu)t}),
\]
\[
H_{1,1}(t) = \frac{\mu}{(\lambda + \mu)^2}((\lambda + \mu)t + 1 - e^{-(\lambda+\mu)t}).
\]