ASSEMBLY SYSTEM.

- The supply chain consists of $N$ nodes, numbered 1 thru $N$, arranged as shown in Figure 6.
- Node $N$ receives items from nodes 1 thru $N-1$ and assembles them into a final product.
- External demands only occur at node $N$ at times $1, 2, 3,...$
- Back orders are allowed at node $N$ alone.
- $l_i =$ shipping time from node $i$ to node $N$, $1 \leq i \leq N - 1$ (non-negative integers).
- No ordering/setup cost at any node.
- Lead times at node $i$ are zero, $1 \leq i \leq N - 1$.
- $h_i =$ holding cost at node $i$.
- $c_i =$ cost of an item from node $i$.
- $p =$ cost of a backordered assembly at the end of a period.
- Push System. Decision variables: how much to order and how much to ship at node $i$, $1 \leq i \leq N - 1$.

OBJECTIVE.

Determine the ordering and shipping policies to minimize the long run rate at which shipping, holding and backordering costs are incurred.
ANALYSIS.

- Renumber the nodes 1 thru \( N - 1 \) so that \( l_1 \geq l_2 \geq \cdots \geq l_{N-1} \).

- Since the lead times and ordering costs at each of these nodes are zero, it makes sense not to carry any inventory of item \( i \) at node \( i \). As soon as a shipping order arrives, the required amount can be produced and shipped off.

- Imagine a sequential assembly process where node 1 ships an item to node 2, where it is combined with an item from node 2 and shipped to node 3, ..., all the way to node \( N \).

- This creates a series system that we have already studied. The nodes in the series system are numbered from 1 to \( N \), node 1 being the top, and \( N \) being the bottom.

- The holding and procurement costs at node \( i \) in this equivalent system are the same as those at node \( i \) in the original system.

- The lead time at node 1 is zero, while that at node \( i \) (\( 2 \leq i \leq N \)) is \( L_i = l_{i-1} - l_i \). (\( l_N = 0 \).) See Figure 7.
• $X_i^m = \text{all units of item } i \text{ in the system plus plus all units ordered in the last } m \text{ periods.}$

• Assume that the initial inventory satisfies:

$$X_i^m \leq X_{i-1}^m \quad i = 2, 3, \ldots, N; \quad m = 0, 1, \ldots, l_i$$

**MAIN RESULT.**

If the initial inventory levels satisfy the above condition, the optimal shipping policy for the original system is the same as the optimal policy for the series system (which is a modified base-stock policy) and can be computed as described earlier.

**References**


DISTRIBUTION SYSTEM.

- The supply chain consists of $J + 1$ nodes, numbered 0 thru $J$, arranged as shown in Figure 8.
- Node 0 receives items from outside, and ships them to nodes 1 thru $J$.
- External demands only occur at nodes 1 thru $J$ at times 1, 2, 3, ...
- Back orders are allowed at nodes 1 thru $J$.
- $l_j = \text{shipping time from node 0 to node } j, 1 \leq i \leq J$ (non-negative integers).
- $L = \text{lead time for orders at node 0}$.
- $K = \text{ordering/setup cost at node 0}$.
- Shipping costs are linear.
- $h_j = \text{holding cost per unit time per item at node } j$.
- $c_0 = \text{procurement cost per item at node 0}$.
- $c_j = \text{cost of shipping an item to node } j \text{ from node 0}, 1 \leq j \leq J$.
- $p_j = \text{cost of a backordered demand at the end of a period at node } j, 1 \leq j \leq J$.
- Push System. Decision variables: how much to order at node 0 and how much to ship to node $j, 1 \leq j \leq J$. 
OBJECTIVE.

Determine the ordering and shipping policies to minimize the long run rate at which ordering, shipping, holding and backordering costs are incurred.
ANALYSIS: NO CENTRALIZED INVENTORY.

- Node zero acts as a centralized ordering and allocation station.
- $T$ = finite horizon length.
- $y_t$ = size of the order placed at node 0 at the beginning of period $t$, $0 \leq t \leq T$.
- $y^t = [y_{t-L} \ldots y_{t-1}]$.
- $x_{jt}$ = inventory position at node $j$ at the beginning of period $t$, $0 \leq t \leq T$.
- $x^t = [x_{1t} \ x_{2t} \ldots \ x_{Jt}]$.
- $z_{jt}$ = allocation to location $j$ in period $t$.
- $z^t = [z_{1t} \ z_{2t} \ldots \ z_{Jt}]$.
- $u_{jt}$ = demand at node $j$ at time $t$.
- $u^t = [u_{1t} \ u_{2t} \ldots \ u_{Jt}]$.
- $U_{jt} = \sum_{s=t}^{t+l_j} u_{js}$.
SYSTEM DYNAMICS.

- The state of the system at the beginning of period \( t \) is given by \((x^t, y^t)\). This is a \( J + L \) dimensional vector.
- System constraints:
  \[
y_t \geq 0, \quad z_{jt} \geq 0, \quad j = 1, 2, \ldots, J,
  \]
  \[
  \sum_{j=1}^{J} z_{jt} = y_{t-L}.
  \]
- System dynamics:
  \[
x^{t+1} = x^t + z^t - u^t
  \]
  \[
y^{t+1} = [y_{t-L+1} \ldots y_t].
  \]
- The holding and shortage costs incurred in period \( t \) are charged in period \( t - l_j \).
  \[
  q_{jt}(x) = h_j E[(x-U_{jt})^+] + p_j E[(U_{jt} - x)^+] \quad \text{if } t \leq T - l_j.
  \]
  \[
  q_{jt}(x) = 0 \quad \text{if } t > T - l_j.
  \]
DYNAMIC PROGRAM.

- \( f_t(x, y) = \) minimum total expected cost from periods \( t \) thru \( T \) starting in state \( x^t = x = (x_1, x_2, ..., x_J) \) and \( y^t = y = (y_1, y_2, ..., y_L) \).
- Decision variables: \( a = \) amount ordered at node 0, \( z = (z_1, ..., z_J) \), \( z_j = \) amount allocated to node \( j \).
- DP recursion:
  \[
  f_T(x, y) = 0,
  \]
  \[
  f_t(x, y) = \min_{a, z} \{ K \delta(a) + c_0 a + \sum_{j=1}^{J} q_{jt}(x_j + z_j) \\
  + E(f_{t+1}[(x + z - u), (y_2, y_3, ..., y_L, a)]) \}
  \]
  The minimum is taken over all \( (a, z) \) satisfying the system constraints:
  \[
  a \geq 0, \quad z_j \geq 0, \quad \sum_{j=1}^{J} z_j = y_1.
  \]
- The state space is \( J + L \) dimensional. No decomposition possible!
- Exact solution is impractical.
APPROXIMATION BY RELAXATION.

• Ignore the non-negativity part of the system constraints. The resulting value function provides an lower bound on the real value function.

• Assume that the parameters $h_j$, $p_j$ and $c_j$ are independent of $j$.

• Consequences (can be proved by induction):
  1. Value function depends on $x$ only via its sum, i.e., the system wide inventory position.
  2. Myopic policy is optimal in each period, i.e., it is optimal to choose $z$ to minimize the one step cost.

• References:
APPROXIMATION BY RESTRICTION.

- Consider policies that base their decisions only on systemwide inventory position.
- Further consider either base-stock or \((s, S)\) policies based on systemwide position.
ANALYSIS: CENTRALIZED INVENTORY ALLOWED.

- Inventories may be kept at node zero.
- Consider infinite horizon stationary problem.
- Three decision variables:
  1. Size of the order to be placed at node 0,
  2. amount to be withdrawn from the central inventory for shipping,
  3. allocation of this amount among the retail nodes.
- One can develop similar DPs, and find approximate solutions by relaxation or restriction approach.