Serial Supply Chain:
PULL System.

- $N =$ number of facilities.
- Facility 1 is the manufacturer, facility $N$ is the retailer.
- Facility $i$ supplies facility $i + 1$, $1 \leq i \leq N - 1$.
- Facility $N$ faces external demand.
- Continuous review policy followed at each facility.
- Back ordering is allowed at all facilities.
- Lead times are facility dependent, iid random variables.
- $h_i =$ holding cost rate per unit at facility $i$. (Increasing function of $i$.)
- $p_i =$ shortage cost rate per unit at facility $i$.
- $K_i =$ ordering cost at facility $i$.
- $c_i =$ Procurement cost at facility $i$.
- PULL System: Each facility follows a continuous review policy and decides to place an order from its supplier based on its inventory status.
OBJECTIVE

Find the optimal ordering policies at each facility so as to minimize the long run cost rate.

APPROXIMATION BY RESTRICTION

Assume that the ordering policy at $i$ facility is of $(s_i, S_i)$ type. Choose the parameters to minimize the long run total cost rate for the entire system.
ANALYSIS FOR $N = 2$: NOTATION.

- One warehouse (facility 0), one retailer (facility 1).
- Location $i$ uses $(S_i - 1, S_i)$ policy. Backorders are allowed at both places.
- $L_i =$ transportation time for the $i$th order from the warehouse to the retailer. Iid random variables with mean $\tau_1$.
- $l_i =$ Lead time for the $i$ order from the warehouse. Iid random variables with man $\tau_0$.
- External demand is $PP(\lambda)$ at the retailer.
- $X_i(t) =$ inventory position at facility $i$ at time $t$.
- $B_i(t) =$ number of backorders at facility $i$ at time $t = \max(0, -X_i(t))$.
- $I_i(t) =$ inventory on hand at facility $i$ at time $t = \max(0, X_i(t))$. 
ANALYSIS AT THE WAREHOUSE.

- Orders arrive according to a PP(\(\lambda\)) at the warehouse.
- \(\{X_0(t), t \geq 0\}\) is the queue length process of an \(M|G|\infty\) queue with arrival rate \(\lambda\) and mean service time \(\tau_0\).
- The limiting distribution: \(X_0 \sim P(\lambda \tau_0)\).
- Expected inventory on hand at the warehouse in steady state:
  \[
  E(I_0) = \sum_{j=0}^{S_0} (S_0 - j)e^{-\lambda \tau_0} \frac{(\lambda \tau_0)^j}{j!}.
  \]
- Expected number of backorders in steady state:
  \[
  E(B_0) = \sum_{j=S_0+1}^{\infty} (j - S_0)e^{-\lambda \tau_0} \frac{(\lambda \tau_0)^j}{j!}.
  \]
ANALYSIS AT THE RETAILER.

- Orders arrive according to a PP$(\lambda)$ at the retailer.
- The expected amount of time an order has to wait at the warehouse before it is shipped from there:
  \[ E(W_0) = E(B_0)/\lambda. \]
- Expected effective lead time for an order placed at the retailer:
  \[ \bar{\tau}_1 = \tau_1 + E(W_0). \]
- Unfortunately the successive lead times are not iid. We approximate the analysis of the retailer inventory position by assuming that they are.
- \( \{X_1(t), t \geq 0\} \) is the queue length process of an \( M|G|\infty \) queue with arrival rate \( \lambda \) and mean service time \( \bar{\tau}_1 \).
- The limiting distribution: \( X_1 \sim P(\lambda \bar{\tau}_1) \).
- Expected inventory on hand at the warehouse in steady state:
  \[ E(I_1) = \sum_{j=0}^{S_1} (S_1 - j)e^{-\lambda \bar{\tau}_1} \frac{(\lambda \bar{\tau}_1)^j}{j!}. \]
- Expected number of backorders in steady state:
  \[ E(B_1) = \sum_{j=S_1+1}^{\infty} (j - S_1)e^{-\lambda \bar{\tau}_1} \frac{(\lambda \bar{\tau}_1)^j}{j!}. \]
COST ANALYSIS.

• Ordering cost rate:
  \[ \lambda(K_0 + K_1). \]

• Procurement cost rate:
  \[ \lambda(c_0 + c_1). \]

• Holding cost rate:
  \[ \sum_{i=0}^{1} h_i E(I_i). \]

• Backordering cost rate:
  \[ \sum_{i=0}^{1} p_i E(B_i). \]

• The long run average total cost is given by
  \[ C(S_0, S_1) = \lambda(K_0 + K_1) + \lambda(c_0 + c_1) + \sum_{i=0}^{1} h_i E(I_i) + \sum_{i=0}^{1} p_i E(B_i). \]

• Choose \( S_0 \) and \( S_1 \) to minimize this.

• For a fixed \( S_0 \), \( C \) is convex in \( S_1 \). Suppose the optimal \( S_1 \) is \( S_1^*(S_0) \).

• \( C(S_0, S_1^*(S_0)) \) is not necessarily convex in \( S_0 \). Hence do a total enumeration, or some other intelligent search.
DISTRIBUTION SYSTEM:
PULL POLICY.

- $N + 1 =$ number of facilities.
- Facility 0 is the manufacturer or warehouse, facilities 1 through $N$ are the retailers.
- Facility 0 supplies facility $i$, $1 \leq i \leq N$.
- Retailers face external demands.
- Back ordering is allowed at all facilities.
- Lead times are facility dependent, iid random variables.
- $h_i =$ holding cost rate per unit at facility $i$.
- $p_i =$ shortage cost rate per unit at facility $i$.
- $K_i =$ ordering cost at facility $i$.
- $c_i =$ Procurement cost at facility $i$.
- PULL System: Each facility follows a continuous review policy and decides to place an order from its supplier based on its inventory status.
OBJECTIVE

Find the optimal ordering policies at each facility so as to minimize the long run cost rate.

APPROXIMATION BY RESTRICTION

Assume that the ordering policy at $i$ facility is of $(s_i, S_i)$ type. Choose the parameters to minimize the long run total cost rate for the entire system.
NOTATION.

- Location $i$ uses $(S_i - 1, S_i)$ policy. Backorders are allowed at all places.
- $L_{ij} =$ transportation time for the $i$th order from the warehouse to the $j$th retailer. Iid random variables with mean $\tau_j$. $(1 \leq j \leq N)$.
- $L_{i0} =$ Lead time for the $i$ order from the warehouse. Iid random variables with man $\tau_0$.
- External demand is $\text{PP}(\lambda_j)$ at the $j$th retailer.
- $X_i(t) =$ inventory position at facility $i$ at time $t$.
- $B_i(t) =$ number of backorders at facility $i$ at time $t = \max(0, -X_i(t))$.
- $I_i(t) =$ inventory on hand at facility $i$ at time $t = \max(0, X_i(t))$. 
ANALYSIS AT THE WAREHOUSE.

• Orders arrive according to a PP(λ) at the warehouse, where
  \[ \lambda = \sum_{j=1}^{N} \lambda_j. \]

• \( \{X_0(t), t \geq 0\} \) is the queue length process of an \( M|G|\infty \) queue with arrival rate \( \lambda \) and mean service time \( \tau_0 \).

• The limiting distribution: \( X_0 \sim P(\lambda \tau_0) \).

• Expected inventory on hand at the warehouse in steady state:
  \[ E(I_0) = \sum_{j=0}^{S_0} (S_0 - j) e^{-\lambda \tau_0} \frac{(\lambda \tau_0)^j}{j!}. \]

• Expected number of backorders in steady state:
  \[ E(B_0) = \sum_{j=S_0+1}^{\infty} (j - S_0) e^{-\lambda \tau_0} \frac{(\lambda \tau_0)^j}{j!}. \]
ANALYSIS AT THE RETAILERS.

- Orders arrive according to a $PP(\lambda_j)$ at the $j$th retailer.
- The expected amount of time an order has to wait at the warehouse before it is shipped from there:
  \[ E(W_0) = E(B_0)/\lambda. \]
- Expected effective lead time for an order placed at the retailer:
  \[ \bar{\tau}_j = \tau_j + E(W_0). \]
- Unfortunately the successive lead times are not iid. We approximate the analysis of the retailer inventory position by assuming that they are.
- $\{X_j(t), t \geq 0\}$ is the queue length process of an $M|G|\infty$ queue with arrival rate $\lambda_j$ and mean service time $\bar{\tau}_j$.
- The limiting distribution: $X_j \sim P(\lambda_j\bar{\tau}_j)$.
- Expected inventory on hand at the warehouse in steady state:
  \[ E(I_j) = \sum_{i=0}^{S_j} (S_j - i)e^{-\lambda_j\bar{\tau}_j}\frac{(\lambda_j\bar{\tau}_j)^i}{i!}. \]
- Expected number of backorders in steady state:
  \[ E(B_j) = \sum_{i=S_j+1}^{\infty} (i - S_j)e^{-\lambda_j\bar{\tau}_j}\frac{(\lambda_j\bar{\tau}_j)^i}{i!}. \]
COST ANALYSIS.

- Ordering cost rate:
  \[ \lambda K_0 + \sum_{j=1}^{N} \lambda_j K_j. \]

- Procurement cost rate:
  \[ \lambda c_0 + \sum_{j=1}^{N} \lambda_j c_j. \]

- Holding cost rate:
  \[ \sum_{i=0}^{N} h_i E(I_i). \]

- Backordering cost rate:
  \[ \sum_{i=0}^{N} p_i E(B_i). \]

- Let \( S = [S_1, \ldots, S_N] \). The long run average total cost is given by
  \[ C(S_0, S) = \lambda K_0 + \sum_{j=1}^{N} \lambda_j K_j + \lambda c_0 + \sum_{j=1}^{N} \lambda_j c_j + \sum_{i=0}^{N} h_i E(I_i) + \sum_{i=0}^{N} p_i E(B_i). \]

- Choose \( S_j \) (\( 0 \leq j \leq N \)) to minimize this.

- For a fixed \( S_0 \), \( C \) is convex in \( S_j \) for each \( 1 \leq j \leq N \). Suppose the optimal \( S_j \) is \( S_j^*(S_0) \).

- \( C(S_0, S^*(S_0)) \) is not necessarily convex in \( S_0 \). Hence do a total enumeration, or some other intelligent search.
ASSEMBLY SYSTEM:
PULL POLICY.

- $N + 1 =$ number of facilities.
- Facility 0 is the assembly unit, facilities 1 through $N$ are the suppliers.
- Each unit assembled at the assembly plant requires one unit from each supplier.
- Facility 0 faces external demands.
- Back ordering is allowed at all facilities.
- Lead times are facility dependent, iid random variables.
- $h_i =$ holding cost rate per unit at facility $i$.
- $p_i =$ shortage cost rate per unit at facility $i$.
- $K_i =$ ordering cost at facility $i$.
- $c_i =$ Procurement cost at facility $i$.
- PULL System: Each facility follows a continuous review policy and decides to place an order from its supplier based on its inventory status.
OBJECTIVE

Find the optimal ordering policies at each facility so as to minimize the the long run cost rate.

APPROXIMATION BY RESTRICTION

Assume that the ordering policy at $i$ facility is of $(s_i, S_i)$ type. Choose the parameters to minimize the long run total cost rate for the entire system.
NOTATION.

- Location $i$ uses $(S_i - 1, S_i)$ policy. Backorders are allowed at all places.
- $L_{ij} =$ lead time for the $i$th order from the $j$th supplier. Iid random variables with mean $\tau_j$. ($1 \leq j \leq N$).
- $T_{ij} =$ Transportation time for the $i$th order from the $j$th supplier to the assembly plant. Iid random variables with mean $\alpha_j$.
- External demand is PP$(\lambda)$ at the assembly plant.
- $X_i(t) =$ inventory position at facility $i$ at time $t$.
- $B_i(t) =$ number of backorders at facility $i$ at time $t = \max(0, -X_i(t))$.
- $I_i(t) =$ inventory on hand at facility $i$ at time $t = \max(0, X_i(t))$. 


ANALYSIS AT A SUPPLIER.

• Orders arrive according to a PP(\(\lambda\)) at the \(j\)th supplier.
• \(\{X_j(t), t \geq 0\}\) is the queue length process of an \(M|G|\infty\) queue with arrival rate \(\lambda\) and mean service time \(\tau_j\).
• The limiting distribution: \(X_j \sim P(\lambda \tau_j)\).
• Expected inventory on hand at the warehouse in steady state:
  \[
  E(I_j) = \sum_{i=0}^{S_j} (S_j - i)e^{-\lambda \tau_j} \frac{(\lambda \tau_j)^i}{i!}.
  \]
• Expected number of backorders in steady state:
  \[
  E(B_j) = \sum_{i=S_j+1}^{\infty} (i - S_0)e^{-\lambda \tau_j} \frac{(\lambda \tau_j)^i}{i!}.
  \]
ANALYSIS AT THE ASSEMBLY PLANT.

- Orders arrive according to a $\text{PP}(\lambda)$ at the assembly plant.
- Each order at the assembly plant generates one order at each of the suppliers.
- Let $W_j$ be the amount of time an order has to wait at the $j$th supplier before it is shipped from there.
  
  \[ E(W_j) = \frac{E(B_j)}{\lambda}. \]

- Effective lead time for an order placed at the $j$th supplier:
  \[ \max(T_1 + W_1, \ldots, T_N + W_N). \]

- Unfortunately the successive lead times are \textit{not} iid, and their mean ($\bar{\tau}_0$) is hard to compute. We approximate the analysis of the retailer inventory position by assuming that they are.

- $\{X_0(t), t \geq 0\}$ is the queue length process of an $M|G|\infty$ queue with arrival rate $\lambda$ and mean service time $\bar{\tau}_0$.

- The limiting distribution: $X_0 \sim P(\lambda \bar{\tau}_0)$.

- Expected inventory on hand and the expected number of backorders in steady state can be computed as before.
COST ANALYSIS.

• Ordering cost rate:
  \[ \lambda(K_0 + \sum_{j=1}^{N} K_j). \]

• Procurement cost rate:
  \[ \lambda(c_0 + \sum_{j=1}^{N} c_j). \]

• Holding cost rate:
  \[ \sum_{i=0}^{N} h_i E(I_i). \]

• Backordering cost rate:
  \[ \sum_{i=0}^{N} p_i E(B_i). \]

• Let \( S = [S_1, \ldots, S_N] \). The long run average total cost is given by
  \[ C(S_0, S) = \lambda(K_0 + \sum_{j=1}^{N} K_j) + \lambda(c_0 + \sum_{j=1}^{N} c_j) + \sum_{i=0}^{N} h_i E(I_i) + \sum_{i=0}^{N} p_i E(B_i). \]

• Choose \( S_j \) (\( 0 \leq j \leq N \)) to minimize this.

• For a fixed \( S_0 \), \( C \) is convex in \( S_j \) for each \( 1 \leq j \leq N \). Suppose the optimal \( S_j \) is \( S_j^*(S_0) \).

• \( C(S_0, S^*(S_0)) \) is not necessarily convex in \( S_0 \). Hence do a total enumeration, or some other intelligent search.