Deterministic
Non-stationary Demand

Assumptions and Notation

- Periodic review: inventory decisions are done at times $t = 1, 2, \ldots, T$.
- $d_t$ = demand at the beginning of period $t$.
- $h_t(x) = \text{cost of holding } x \text{ items in period } t$. Concave function of $x$. Example:
  \[ h_t(x) = hx. \]
- $c_t(y) = \text{cost of ordering } y \text{ items in period } t$. Concave function of $y$. Example:
  \[ c_t(y) = K(1 - \delta_{y,0}) + cy. \]

- No shortages permitted.
- Zero lead time.
- Zero initial inventory.

Objective
Decide how much to order and when to order in order to minimize the total cost.
ANALYSIS

- Sequence of events: Order placed, Procurement cost incurred, Order received, Demand observed, Demand satisfied, Holding cost incurred on the remaining inventory.

- $x_t$ = ending inventory in period $t$. We start period 1 with $x_0 = 0$.

- $y_t$ = order quantity in period $t$.

- $x_t = x_{t-1} + y_t - d_t, \quad t = 1, \ldots, T$.

- Total cost:

\[ TC = \sum_{t=1}^{T} [c_t(y_t) + h_t(x_t)]. \]

- Math Program:

Minimize \[ \sum_{t=1}^{T} [c_t(y_t) + h_t(x_t)] \]
Subject to

- $x_t = x_{t-1} + y_t - d_t, \quad t = 1, \ldots, T$,
- $x_t, y_t \geq 0, \quad t = 1, 2, \ldots, T$,
- $x_0 = 0$. 

2
Wagner-Whitin Algorithm


• The optimal value of $y_t$ is in the set

$$\{0, \ d_t, \ d_t + d_{t+1}, \ \cdots, \ d_t + \cdots + d_T\}.$$ 

• An optimal policy has the property

$$y_t x_{t-1} = 0.$$ 

• It is optimal to order only when the starting inventory is zero.

• An optimal policy is obtained by finding the shortest path in a network with $T + 1$ nodes labeled 1, 2, ..., $T + 1$, and with arc length $c_{i,j}$ between nodes $1 \leq i < j \leq T + 1$ given by

$$c_{i,j} = c_i \left( \sum_{k=i}^{j-1} d_k \right) + h_k \left( \sum_{l=k+1}^{j-1} d_l \right).$$ 

• Let $f_i$ be the cost of following an optimal policy when $i$ periods remain. Dynamic Programming recursion:

$$f_{T+1} = 0$$

$$f_i = \min_{j=i+1, \ldots, T+1} \{c_{i,j} + f_j\}, \quad i = 1, 2, \ldots, T.$$
Forward Algorithm

• $g_i$ = minimum policy cost for period $1, 2, \ldots, i$, given that the inventory level is zero at the end of period $i$, $0 \leq i \leq T$.

• $g_i$ can be computed in a forward recursive fashion as follows:

$$g_0 = 0,$$

$$g_j = \min_{i=1,2,\ldots,j} \{g_{i-1} + c_{i,j+1}\}, \quad i = 1, 2, \ldots, T.$$

• Let $i_j$ be the value that minimizes the RHS for $g_j$. Then for the $j$ period problem the last production occurs in period $i_j$ in an optimal policy. Thus the optimal policy can be worked out in a backward way.
Example

- \( h_t(x) = h x, \; c_t(y) = K (1 - \delta_{y,0}) + cy. \)
- \( T = 5, \; K = 250, \; c = 2, \; h = 1, \)
  \[ d_1 = 220, \; d_2 = 280, \; d_3 = 360, \; d_4 = 140, \; d_5 = 270. \]
- The \( c_{i,j} \) are as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>690</td>
<td>1530</td>
<td>2970</td>
<td>3670</td>
<td>5290</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>810</td>
<td>1890</td>
<td>2450</td>
<td>3800</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>970</td>
<td>1390</td>
<td>2470</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>530</td>
<td>1340</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>790</td>
<td></td>
</tr>
</tbody>
</table>

- Backward recursion:
  \( f_6 = 0, \; f_5 = 790, \; j_5 = 5, \; f_4 = 1320, \; j_4 = 4, \)
  \( f_3 = 2180, \; j_3 = 4, \; f_2 = 2990, \; j_2 = 2, \; f_1 = 3680, \; j_1 = 1. \)
- Optimal production:
  \( y_1 = 220, \; y_2 = 280, \; y_3 = 500, \; y_4 = 0, \; y_5 = 270. \)
- Optimal cost = $3680.
- Forward recursion:
  \( g_0 = 0, \; g_1 = 690, \; i_1 = 1, \; g_2 = 1500, \; i_2 = 2, \)
  \( g_3 = 2470, \; i_3 = 3, \; g_4 = 2890, \; i_4 = 3, \; g_5 = 3680, \; i_5 = 5. \)
- Produces the same optimal production schedule, as it must.
Planning Horizons

• $l(t)$: the last period when production occurs in an optimal $t$ period problem.

• $l(t)$ is a non-decreasing function of $t$ (consequence of concave costs).

• $l(t) = t$ implies that the optimal policy for periods $1, 2, ..., t-1$ does not depend upon data for periods $t$ onwards.

• If $l(t) = t$, we say that the planning horizon is $1, 2, ..., t - 1$.

• In the previous example, $l(1) = 1$, $l(2) = 2$, $l(3) = 3$, $l(4) = 3$, $l(5) = 5$. 
Silver-Meal Heuristics


- \( C(T) \) = average cost of holding and setup per period if the current order satisfies the demand for the next \( T \) periods.

\[
C(T) = [K + hd_2 + 2hd_3 + \ldots + (T - 1)hd_T]/T.
\]

- Compute \( C(T) \) for \( T = 1, 2, \ldots \), and stop as soon as \( C(T) > C(T - 1) \), and order \( d_1 + d_2 + \ldots + d_{T-1} \) in the first period.

- Example: \( K = 250, \ h = 1, \ d = [220, 280, 360, 140, 270] \).

- Step 1: \( C(1) = 250, \ C(2) = (250 + 280)/2 = 265 \). Hence \( y_1 = 220 \).

- Step 2: \( C(1) = 250, \ C(2) = (250 + 360)/2 = 305 \). Hence \( y_2 = 280 \).

- Step 3: \( C(1) = 250, \ C(2) = (250 + 140)/2 = 195, \ C(3) = (250 + 140 + 270)/3 = 220 \). Hence \( y_3 = 360 + 140 = 500 \).

- Step 4: \( y_4 = 0, y_5 = 270 \).
Extensions

- Non-zero lead times.
- Production Capacities: $y_t \leq U_t$, where $U_t$ is the upper bound on the production capacity in period $t$. Reference: Florian, M., and M. Klein (1971). Deterministic production planning with concave costs and capacity constraints. Management Science 18, 12-20.