Parallel Supply Chain: Assembly System

Assumptions and Notation

• See Fig 4.2.
• The supply chain consists of $N + 1$ stations.
• Station $i$ manufactures the items, $1 \leq i \leq N$.
• Station 0 assembles one unit from each of the stations 1 through $N$ to produce one finished item.
• Station 0 faces the external demand at a constant rate $\lambda$.
• $K_i =$ ordering/setup cost at station $i$, $0 \leq i \leq N$.
• Lead time =0.
• $h_i' > 0 =$ holding cost per item held at station $i$ per unit time.
• No shortages permitted.

Objective

Find the optimal reordering and production policies at the $N + 1$ stations so as to minimize the long run cost
per unit time.

Analysis

• $x_i(t) = \text{inventory on hand at station } i \text{ at time } t.$
  $x_i(0) = 0 \text{ for all } i.$

• $n_i(t) = \text{number of reorders/setups at station } i \text{ upto time } t.$

• Long run average cost per unit time

$$\lim_{T \to \infty} \frac{1}{T} \left[ \sum_{i=0}^{n} n_i(T)K_i + \int_{0}^{T} h'_i x_i(t)dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{T} \left[ \sum_{i=0}^{n} n_i(T)K_i + \int_{0}^{T} h_i X_i(t)dt \right]$$

where $h_i = h'_i$ is the echelon holding cost rate, and $X_i(t) = x_i(t) + x_0(t)$ is the echelon inventory at station $i$, $1 \leq i \leq N$; and $h_0 = h'_0 - \sum_{i=1}^{N} h'_i$ and $X_0(t) = x_0(t)$.

• A policy is called stationary if the times between consecutive reorder points at station $i$ is a constant, say $T_i$, i.e., if $n_i(t) = \left[ \frac{t}{T_i} \right]$, for all $t$ and all $i$.

• A stationary policy is called nested if $T_i = m_i T_0$ for some positive integer $m_i$, for $i = 2, 3, ..., N$.

• Result 1: For an $N + 1$ station assembly system it is optimal to follow a stationary, nested policy if $h_0 > 0$.

• Result 2: In a stationary nested optimal policy described by $T_0, T_1, \cdots, T_N$, $x_i(T_i^-) = 0$. 
Centralized Control

- The global optimization problem is:

  \[
  \text{Minimize} \quad \sum_{i=0}^{N} \left[ \frac{K_i}{T_i} + g_i T_i \right] \\
  \text{Subject to:} \quad T_i = m_i T_0, \quad 1 \leq i \leq N, \\
  \quad m_i \geq 1, \quad \text{integer} \\
  \quad T_0 \geq 0.
  \]

- The relaxed problem:

  \[
  \text{Minimize} \quad \frac{K_0}{T_0} + g_0 T_0 + \sum_{i=1}^{N} \left[ \frac{K_i}{m_i T_0} + g_i m_i T_0 \right] \\
  \text{Subject to:} \quad m_i \geq 0, \quad 1 \leq i \leq N, \\
  \quad T_0 \geq 0.
  \]
Solution to the Relaxed Problem

- Fix $m = [m_1, m_2, \ldots, m_N]$. The objective function:
  \[
  \frac{K_0 + \sum_{i=1}^{N} K_i/m_i}{T_0} + (g_0 + \sum_{i=1}^{N} g_i m_i)T_0.
  \]

- Optimal solution:
  \[
  T_0^*(m) = \sqrt{\frac{K_0 + \sum_{i=1}^{N} K_i/m_i}{g_0 + \sum_{i=1}^{N} g_i m_i}}.
  \]

- Optimal cost:
  \[
  C^*(m) = 2 \sqrt{(K_0 + \sum_{i=1}^{N} K_i/m_i)(g_0 + \sum_{i=1}^{N} g_i m_i)}.
  \]

- To find optimal $m$, set
  \[
  \frac{dC^*(m)}{dm_i} = 0
  \]
  which yields
  \[
  m_i = \sqrt{\frac{K_i}{g_i}} \cdot \sqrt{\frac{g_0 + \sum_{i=1}^{N} g_i m_i}{K_0 + \sum_{i=1}^{N} K_i/m_i}} = \sqrt{\frac{K_i}{g_i}} \cdot A.
  \]
• The objective function becomes

\[ C^*(m) = C^*(A) = 2 \sqrt{K_0 + \sum_{i=1}^{N} \sqrt{K_i g_i} / A}(g_0 + \sum_{i=1}^{N} \sqrt{K_i g_i} A). \]

\( A \) is given by

\[ A = \sqrt{g_0/K_0}. \]

• Optimal \( m \):

\[ m_i^* = \frac{\sqrt{K_i/g_i}}{\sqrt{K_0/g_0}}, \quad 1 \leq i \leq N. \]

• The optimal cost

\[ C^* = 2 \sum_{i=0}^{N} \sqrt{K_i g_i}. \]

• The optimal reorder intervals:

\[ T_{i^*} = \sqrt{K_i/g_i}, \quad 0 \leq i \leq N. \]
Solution to the Original Problem

- Set
  \[ T_0^* = \sqrt{K_0/g_0}. \]

- Set
  \[ m_i^* = \left\lfloor \frac{\sqrt{K_i/g_i}}{\sqrt{K_0/g_0}} \right\rfloor, \quad 1 \leq i \leq N. \]
  \[ T_i^* = m_i^* T_0^*, \quad 1 \leq i \leq N. \]

- One can also choose a power of two solution, leading to the 6\% bound on optimal value function.
Decentralized Control

- Optimal order interval at the assembly plant:
  \[ T_0^* = \sqrt{K_0/g_0'} \leq \sqrt{K_0/g_0}. \]

- Optimal order interval at station \( i \):
  \[ T_i^* = \sqrt{K_i/g_i'} = \sqrt{K_i/g_i}, \quad 1 \leq i \leq N. \]
Parallel Supply Chain:
Distribution System
Assumptions and Notation

- The supply chain consists of \( N + 1 \) stations.
- Station 0 is the central warehouse.
- Station \( i \) is the \( i \)th retail station, \( 1 \leq i \leq N \).
- Station \( i \) faces the external demand at a constant rate \( \lambda_i \), \( 1 \leq i \leq N \).
- \( K_i \) = ordering/setup cost at station \( i \), \( 0 \leq i \leq N \).
- Lead time = 0.
- \( h_i' > 0 \) = holding cost per item held at station \( i \) per unit time.
- No shortages permitted.

Objective

Find the optimal reordering and production policies at the \( N + 1 \) stations so as to minimize the long run cost per unit time.

Central Control: Analysis

- The optimal policies need not be stationary, nor nested.

- The optimal policies have Zero-ordering property, i.e., the inventory at a station $i$ is zero whenever an order is placed at that station.

- A stationary policy with zero-ordering property is described by reordering intervals $[T_0, T_1, \cdots, T_N]$. It is called an integer-ratio policy if $T_i = m_i T$, where $T > 0$ is a base period and $m_i$ are positive integers.

- We attempt to find an optimal policy within the set of stationary, zero-ordering, integer-ratio policies.

- $h_i = h'_i - h_0 = \text{echelon holding cost at station } i \text{ for } 1 \leq i \leq N$. $h_0 = h'_0$. 
Central Control: Global Optimization

- Let \( g_i = \frac{1}{2} h_i \lambda_i \), and \( g^i = \frac{1}{2} h_0 \lambda_i \).
- The global optimization problem is:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=0}^{N} \frac{K_i}{T_i} + \sum_{i=1}^{N} [g_i T_i + g^i \max(T_0, T_i)] \\
\text{Subject to:} & \quad T_i = m_i T, \quad 0 \leq i \leq N, \\
& \quad m_i \geq 1, \quad \text{integer} \\
& \quad T \geq 0.
\end{align*}
\]

- The relaxed problem:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=0}^{N} \frac{K_i}{m_i T} + \sum_{i=1}^{N} [g_i m_i T + g^i \max(m_0, m_i) T] \\
\text{Subject to:} & \quad m_i \geq 0, \quad 1 \leq i \leq N, \\
& \quad T \geq 0.
\end{align*}
\]
Central Control: Algorithm for the relaxed problem.

Step 1. Compute \( \tau'_i = \sqrt{2K_i/\lambda h_i} \) (conventional) and \( \tau_i = \sqrt{2K_i/\lambda h_i} \) (echelon), for \( i = 1, 2, \ldots N \). Sort them into a non-decreasing sequence of \( 2N \) numbers.

Step 2. Set \( E = G = \phi, \ L = \{1, 2, \ldots, N\}, \ K = K_0, \ H = \sum_{i=1}^{N} g^i \).

Step 3. Let \( \tau \) be the largest element of \( S \).
If \( \tau^2 \geq K/H \) and \( \tau = \tau_i \),
\[
S = S - \{ \tau \}, \quad E = E \cup \{ i \}, \quad L = L - \{ i \}, \quad K = K + K_i, \quad H = H + g_i.
\]
Go to Step 3.
If \( \tau^2 \geq K/H \) and \( \tau = \tau'_i \),
\[
S = S - \{ \tau \}, \quad E = E - \{ i \}, \quad G = G \cup \{ i \}, \quad K = K - K_i, \quad H = H - g_i - g^i.
\]
Go to Step 3.

Step 4 Current \( G, L, E \), are optimal. Let \( T_0^* = \sqrt{K/H} \).
\[
T_i^* = T_0^*, \quad i \in E,
\]
\[
T_i^* = \sqrt{K_i/g_i} < T_0^*, \quad i \in L,
\]
\[
T_i^* = \sqrt{K_i/(g_i + g^i)} > T_0^*, \quad i \in G.
\]