RENEWAL DEMANDS.

- Demands occur one at a time according to a renewal process.
- Multiple outstanding orders allowed.
- Inventory Position = Inventory on hand - Inventory on backorder + Inventory on order.
- Order of size \( Q = S - s \) is placed whenever the inventory position falls to \( s \).
- \( T = \) fixed lead time, i.e. order placed at time \( t \) will appear at time \( t + T \). Independent of order size.
- \( K = \) ordering cost.
- \( h = \) holding cost.
- \( p = \) cost of a back ordered demand per item per unit time.

OBJECTIVE.

Determine \( S \) and \( s \) to minimize the long run rate at which ordering, holding and back ordering costs are incurred.
ANALYSIS.

• \( D(t) = \) demand over \((0, t] \).
• \( D(u, t) = \) demand over \((u, t], t \geq u \).
• Assumption: \( \{D(t), t \geq 0\} \): A renewal process with common inter renewal time cdf \( F(\cdot) \) and mean \( \tau \).
• \( \{D(u, u + t), t \geq 0\} \): A delayed renewal process.
• As \( u \to \infty \), \( \{D(u, u + t), t \geq 0\} \) tends to an equilibrium renewal process.
• \( V(t) = \) Inventory on hand - inventory on backorder at time \( t \).
• Inventory on hand at time \( t = \max\{0, V(t)\} \).
• Inventory on backorder at time \( t = \max\{0, -V(t)\} \).
• \( I(t) = \) Inventory position at time \( t \).
• \( I(t) \in \{s + 1, s + 2, ..., S\} \).
• \( V(t) = I(t - T) - D(t - T, t), \) for \( t \geq T \).
• \( I(t) \) and \( D(t, t + u) \) are independent if \( t \) is a demand arrival period, and also in the limit as \( t \to \infty \).
• Hence we study \( \{I(t), t \geq 0\} \) first.
LIMITING BEHAVIOR OF $I(t)$.

- $\{I(t), t \geq 0\}$ is a regenerative process on $\{s + 1, s + 2, ..., S\}$.
- $I$ regenerates whenever it hits $S$.
- Expected cycle length $= (S - s)\tau$.
- Expected time spent in state $j$ during a cycle $= \tau$, for $s + 1 \leq j \leq S$.
- Assume $F$ is aperiodic. Then $I(t)$ has a limiting distribution.
- $p_j = \lim_{t \to \infty} P(I(t) = j) = \frac{\tau}{(S - s)\tau} = \frac{1}{S - s}$.

Thus the inventory position is uniformly distributed over $\{s + 1, ..., S\}$.
LIMITING BEHAVIOR OF $V(t)$.

- As $t \to \infty$, the distribution of $D(t-T, t)$ approaches that of $D_e(T)$, where $\{D_e(t), t \geq 0\}$ is the equilibrium renewal process.
- $F_e(t) = \frac{1}{T} \int_0^t (1 - F(u))du$.
- $F_e \ast F(t) = \int_0^t F_e(t-u) dF(u)$ is the convolution of $F_e$ and $F$.
- $F^{*k}(t)$ is the $k$-fold convolution of $F$ with itself.
- $q_0 = P(D_e(T) = 0) = 1 - F_e(T)$.
- $q_j = P(D_e(T) = j) = F_e \ast F^{*(j-1)}(T) - F_e \ast F^*(T)$.
- The limiting distribution of $V$ can now be computed as a convolution of two independent random variables:
  \[ V(\infty) = I(\infty) - D_e(T). \]
- Thus, for $j \leq S$:
  \[ \pi_j = P(V(\infty) = j) = \sum_{i=s+1}^{S} \frac{1}{S} q_{i-j}, \]
  where we use $q_j = 0$ if $j < 0$. 

4
LONG RUN COST RATE.

- Ordering cost rate = \( \frac{K}{(S-s)\tau} \).
- Holding cost rate = \( \sum_{j=0}^{S} jh\pi_j \).
- Backorder cost rate = \( \sum_{j=1}^{\infty} pj\pi_{-j} \).
- Total cost rate =

\[
C(S, s) = \frac{K}{(S-s)\tau} + \sum_{j=0}^{S} jh\pi_j + \sum_{j=1}^{\infty} pj\pi_{-j}.
\]

- Numerically compute \( S \) and \( s < S \) that minimize the above cost rate function.
BATCH-RENEWAL DEMANDS.

- The demands occur one batch at a time.
- The batch sizes are iid with common pdf $\phi$, and cdf $\Phi$.
- The batches arrive according to a renewal process with common inter renewal time cdf $F(\cdot)$ and mean $\tau$.
- The batch sizes are independent of the batch arrival process.
- Multiple outstanding orders allowed.
- Inventory Position = Inventory on hand - Inventory on backorder + Inventory on order.
- Whenever the inventory position falls below $s$ an order is placed to bring the inventory upto $S$. (The order size may vary.)
- $T$ = fixed lead time, i.e. order placed at time $t$ will appear at time $t + T$. Independent of order size.

OBJECTIVE.

Determine $S$ and $s$ to minimize the long run rate at which ordering, holding and back ordering costs are incurred.
ANALYSIS.

- $D(t) =$ demand over $(0, t]$.
- $D(u, t) =$ demand over $(u, t]$, $t \geq u$.
- Assumption: $\{D(t), t \geq 0\} =$ A renewal reward process with common inter renewal time cdf $F(\cdot)$ and mean $\tau$, and reward density $\phi$.
- $\{D(u, u+t), t \geq 0\} =$ A delayed renewal reward process.
- As $u \to \infty$, $\{D(u, u + t), t \geq 0\}$ tends to an equilibrium renewal reward process.
- $V(t) =$ Inventory on hand - inventory on backorder at time $t$.
- Inventory on hand at time $t = \max\{0, V(t)\}$.
- Inventory on backorder at time $t = \max\{0, -V(t)\}$.
- $I(t) =$ Inventory position at time $t$.
- $I(t) \in [s, S]$.
- $V(t) = I(t - T) - D(t - T, t)$, for $t \geq T$.
- $I(t)$ and $D(t, t + u)$ are independent if $t$ is a demand arrival period, and also in the limit as $t \to \infty$.
- Hence we study $\{I(t), t \geq 0\}$ first.
LIMITING BEHAVIOR OF $I(t)$.

- $\{I(t), t \geq 0\}$ is a regenerative process on $[s, S]$. It regenerates whenever it hits $S$.
- $X_n$ = the $n$th inter-batch arrival time.
- $Y_n$ = size of the $n$th batch.
- $N(t)$ = renewal process generated by $\{Y_n, n \geq 1\}$.
- Let $M(t) = E(N(t))$ be the renewal function of $N$:
  \[
  M(t) = \sum_{j=1}^{\infty} \Phi^*(t).
  \]
- Suppose $I(0) = S$. Order is placed at time
  \[
  \min\{t \geq 0 : D(t) > S - s\} = \sum_{n=1}^{N(S-s)+1} X_n.
  \]
- Result from renewal theory
  \[
  E\left( \sum_{n=1}^{N(S-s)+1} X_n \right) = (1 + M(S - s))\tau.
  \]
- Expected cycle length = $(1 + M(S - s))\tau$.
- $m(x)dx$ = probability that there is a renewal in the interval $(x, x + dx)$.
- Expected time spent in state $(x, x + dx)$ during a cycle = $\tau m(S - x)dx$, for $s \leq x \leq S$.
- Assume $F$ is aperiodic. Then $I(t)$ has a limiting density:
  - $p(x) = \frac{m(S-x)}{1+M(S-s)}$. 

8
LIMITING BEHAVIOR OF $V(t)$.

- As $t \to \infty$, the distribution of $D(t - T, t)$ approaches that of $D_e(T)$, where $\{D_e(t), t \geq 0\}$ is the equilibrium renewal reward process.
- $q(0) = P(D_e(T) = 0) = 1 - F_e(T)$.
- $q(y)dy = P(D_e(T) \in (y, y + dy))$
  \[= \sum_{j=1}^{\infty} (F_e * F^*(j-1)(T) - F_e * F^j(T)) \phi^*(y)dy.\]

Here $\phi^*(y) = 1$ and
\[\phi^j(y) = \int_{0}^{y} \phi^*(j-1)(y - u) \phi(u)dy, \quad j \geq 1.\]

- The limiting distribution of $V$ can now be computed as a convolution of two independent random variables:
  \[V(\infty) = I(\infty) - D_e(T).\]

- Thus, for $x \leq S$:
  \[\pi(x)dx = P(V(\infty) \in (x, x + dx))\]
  \[= p(x)q(0)dx + \int_{y=s}^{S} p(y)q(x - y)dydx,\]
  where we use $q(y) = 0$ if $y < 0$ and $p(x) = 0$ if $x < s$. 

LONG RUN COST RATE.

• Ordering cost rate = \( \frac{K}{1+M(S-s)\tau} \).

• Holding cost rate = \( \int_{x=0}^{S} hx\pi(x)dx \).

• Backorder cost rate = \( \int_{x=0}^{\infty} px\pi(-x)dx \).

• Total cost rate =
  \[ C(S, s) = \frac{K}{1+M(S-s)\tau} + \int_{x=0}^{S} hx\pi(x)dx + \sum_{x=0}^{\infty} px\pi(-x)dx. \]

• Numerically compute \( S \) and \( s < S \) that minimize the above cost rate function.
(S - 1, S) INVENTORY POLICIES
BACKORDER CASE.

- $S =$ stock-up-to level.
- Place an order for a replenishment whenever a demand occurs.
- This produces a $(s, S)$ policy with $s = S - 1$.
- Demands arrive according to a PP($\lambda$).
- The lead times are iid with cdf $G(\cdot)$ and mean $\tau$.
- Orders can cross!
- Back ordering allowed.
- $K =$ ordering cost.
- $h =$ holding cost.
- $p =$ cost of a back ordered demand per item per unit time.

OBJECTIVE.

Determine $S$ to minimize the long run rate at which ordering, holding and back ordering costs are incurred.
ANALYSIS.

- \( W(t) = \) Inventory on order at time \( t \).
- Inventory on hand at time \( t = [S - W(t)]^+ \).
- Inventory on back order at time \( t = [W(t) - S]^+ \).
- \( W(t) \) increases by 1 whenever a demand occurs, and decreases by 1 whenever an order arrives.
- Since the arrival process is PP and lead times are iid, \( W(t) \) can be thought of as the number of customers in an \( M/G/\infty \) system at time \( t \).
- Result from queueing theory: The limiting distribution of \( W(t) \) is \( P(\lambda \tau) \), i.e.,

\[
P(W = k) = e^{-\lambda \tau} \frac{(\lambda \tau)^k}{k!} \quad k \geq 0.
\]
- Ordering cost rate = \( \lambda K \).
- Holding cost rate = \( hE([S - W]^+) \).
- Backordering cost rate = \( pE([W - S]^+) \).
- Total cost rate = \( C(S) = \lambda K + hE([S - W]^+) + pE([W - S]^+) \).
- Determine \( S \) to minimize \( C(S) \).
(S − 1, S) INVENTORY POLICIES
LOST SALES CASE.

• S = stock-up-to level.
• Place an order for a replenishment whenever a demand occurs.
• This produces a (s, S) policy with s = S − 1.
• Demands arrive according to a PP(λ).
• The lead times are iid with cdf G(·) and mean τ.
• Orders can cross!
• Stock outs lead to lost sales.
• K = ordering cost.
• h = holding cost.
• p = cost of a lost sale.

OBJECTIVE.

Determine S to minimize the long run rate at which ordering, holding and lost sales costs are incurred.
ANALYSIS.

- $W(t) = \text{Inventory on order at time } t$.
- Inventory on hand at time $t = S - W(t)$.
- Rate at which sales are lost at time $t = \lambda P(W(t) = S)$.
- $W(t)$ increases by 1 whenever $W(t) < S$ and a demand occurs, and decreases by 1 whenever an order is fulfilled.
- Since the arrival process is PP and lead times are iid, $W(t)$ can be thought of as the number of customers in an $M/G/S/S$ system at time $t$.
- Result from queueing theory: The limiting distribution of $W(t)$ is $P(\lambda \tau)$ truncated at $S$, i.e.,

$$P(W = k) = \frac{e^{-\lambda \tau} (\lambda \tau)^k}{k!} \left/ \sum_{i=0}^{S} \frac{e^{-\lambda \tau} (\lambda \tau)^i}{i!} \right., \quad 0 \leq k \leq S.$$

- Ordering cost rate $= \lambda(1 - P(W = S))K$.
- Holding cost rate $= hE(S - W)$.
- Lost sales cost rate $= p\lambda P(W = S)$.
- Total cost rate $= C(S) = \lambda(1 - P(W = S))K + hE(S - W) + p\lambda P(W = S)$.
- Determine $S$ to minimize $C(S)$.