Section 3.5: Rational Functions

Objectives
Upon completion of this lesson, you will be able to:

- Sketch the graph of a rational function by finding:
  - $x$- and $y$-intercepts,
  - horizontal and vertical asymptotes,
  - possible holes in the graph, and
  - where the function is above or below the x-axis, i.e. where $f(x) > 0$ or $f(x) < 0$.

- Determine the domain/range of a rational function and where the function is increasing/decreasing.

- Given a rational function, answer questions such as, $x \to a, f(x) \to \ldots$, or as $x \to a^+, f(x) \to \ldots$, or as $x \to a^-, f(x) \to \ldots$.

- Given conditions, write the equation of a rational function.

Required Reading
Swokowski/Cole: Section 3.5, pages 249-264.

Discussion

Generalizations for Horizontal Asymptotes

- When the highest power of the numerator exceeds that of the denominator, there is no horizontal asymptote. The function will tend toward infinity as $x$ approaches $\pm \infty$.
  
  Example: $f(x) = \frac{x^3}{x + 5}$.

- When the highest power of the denominator exceeds that of the numerator, the horizontal asymptote is $y = 0$. Example: $f(x) = \frac{x - 1}{x^3 + x}$.

- When the highest power of the numerator is the same as the highest power of the denominator, the horizontal asymptote is the ratio of the coefficients on the highest power terms. Example: $f(x) = \frac{5x^3 - 1}{x - 2x^3}$. The horizontal asymptote is $y = -\frac{5}{2}$.

You are not responsible for the topic of oblique asymptotes in this section.

Holes in Rational Functions: See Example 4 on page 257 of your text.

For a rational function $f(x)$ with a common factor in the numerator and denominator of $(x - a)$ and a reduced version of the function called $F(x)$, there is a hole in the function at the point $(a, F(a))$. 
Example 1: Find the coordinates of the hole in the function \( f(x) = \frac{(x + 2)(x + 5)}{(x + 2)(x + 1)} \).

Solution: \( f(x) = \frac{(x + 2)(x + 5)}{(x + 2)(x + 1)} = \frac{x + 5}{x + 1}, \) where \( x \neq -2 \).

The function \( f(x) \) has a hole at \( x = -2 \) because \((x + 2)\) is a common factor. Using the reduced version of the function, we see that the hole is at \((-2, -\frac{2 + 5}{-2 + 1})\) or \((-2, -3)\).

This section of the text is very well written with excellent examples. Following is one example of identifying the important attributes of a rational function and its graph and one example of writing a rational function given specific attributes.

Example 2: Sketch the graph of \( f(x) = \frac{x + 3}{x(x - 2)} \).

Solution: This function does not have a hole in it since there are no common factors in the numerator and denominator.

- \( x \)-intercept: \( f(x) = 0 \) when \( x + 3 = 0 \). Thus, \( x = -3 \), and \((-3, 0)\) is the \( x \)-intercept.

- Vertical asymptote: \( f(x) \) is undefined when \( x = 0 \) and when \( x - 2 = 0 \). Thus, vertical asymptotes are located at \( x = 0 \) or \( x = 2 \).

- \( y \)-intercept: \( f(x) \) does not have a \( y \)-intercept since the \( y \)-axis is a vertical asymptote.

- Horizontal asymptote: By the theorem on page 266, since the highest power on \( x \) is in the denominator, the horizontal asymptote is \( y = 0 \).

By constructing a number line with the values from the \( x \)-intercept (-3) and the vertical asymptotes (0 and 2), we can see where the function is above and below the \( x \)-axis by checking values in the intervals. Select test numbers in each interval of the number line. We will use -4, -1, 1, and 3. Plug these values into the expression \( \frac{x + 3}{x(x - 2)} \), and indicate a + or - on the number line for the sign of the expression at each test point.

\[ \frac{x + 3}{x(x - 2)} > 0 \) or above the \( x \)-axis from \((-3, 0)\) and \((2, \infty)\).

\[ \frac{x + 3}{x(x - 2)} < 0 \) or below the \( x \)-axis from \((-\infty, -3)\) and \((0, 2)\).
From the sketch of the given function, we can see that:

a. As \( x \to 0^- \), \( f(x) \to +\infty \).

b. As \( x \to 0^+ \), \( f(x) \to -\infty \).

c. As \( x \to 2^- \), \( f(x) \to -\infty \).

d. As \( x \to 2^+ \), \( f(x) \to +\infty \).

Remember that the graph of a rational function may never cross a vertical asymptote, but the graph may cross a horizontal asymptote as in the example above. To determine if the graph of a rational function crosses the horizontal asymptote, set the function equal to the value of the horizontal asymptote. In this case set \( \frac{x+3}{x(x-2)} = 0 \). We see that the function crosses the horizontal axis at \( x = -3 \), the \( x \)-intercept.

**Example 3:** Write an equation of a rational function that satisfies the conditions:
- vertical asymptotes: \( x = -8 \) and \( x = 5 \).
- horizontal asymptote: \( y = 0 \).
- \( x \)-intercept: \((-2, 0)\).
- \( f(0) = -2 \).
- hole at \( x = 3 \).

**Solution:**
With vertical asymptotes at \( x = -8 \) and \( x = 5 \), that gives us factors of \( (x + 8) \) and \( (x - 5) \) in the denominator.
With an \( x \)-intercept of \((-2, 0)\), we have a factor in the numerator of \( (x + 2) \).
With a hole at \( x = 3 \), we have a factor of \( (x - 3) \) in both the numerator and the denominator.

So far, \( f(x) = \frac{(x+2)(x-3)}{(x+8)(x-5)(x-3)} \).
If the factors above are multiplied out, the denominator has the highest power on \( x \). Thus, the horizontal asymptote is \( y = 0 \).
Lastly, compute \( f(0) \). We get \(-1/20\), not \(-2\). We need to multiply \(-1/20\) by 40 to get \(-2\).

Hence, \( f(x) = \frac{40(x + 2)(x - 3)}{(x + 8)(x - 5)(x - 3)} \).

**Practice Problems**
Work these problems. Answers to the odd numbered problems can be found at the end of your text, even answers are below.

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Compare problems 11 and 13. They are the same except for the hole.

Answers to even exercises.

2b. D: all reals except \( x = 0 \).
    R: \((0, \infty)\)

2c. Incr: \((-\infty, 0)\)
    Decr: \((0, \infty)\)

4. VA: \( x = 1 \); HA: \( y = \frac{2}{5} \); hole: \((-2, -\frac{4}{15})\)

6. \( f(x) = \frac{-2(x - 3)(x - 4)}{(x + 1)(x - 4)} \)

46. \( f(x) = \frac{15(x - 2)}{x(x + 2)} \)

48. \( f(x) = \frac{2x(x + 2)(x - 1)}{x(x + 1)(x - 3)} \)